Boson-fermion mappings for odd systems
from supercoherent states

J. Dobaczewski,[*] F. G. Scholtz, and H. B. Geyer

Institute of Theoretical Physics, University of Stellenbosch,

Stellenbosch 7600, South Africa

We extend the formalism whereby boson mappings can be derived from generalized coherent states to boson-fermion mappings for systems with an odd number of fermions. This is accomplished by constructing supercoherent states in terms of both complex and Grassmann variables. In addition to a known mapping for the full so(2N+1) algebra, we also uncover some other formal mappings, together with mappings relevant to collective subspaces.

I. INTRODUCTION

Phenomenological models of collective states in odd fermion systems (mostly nuclei in the present context) usually assume that these states can be approximated by states in the product Hilbert space

$$\mathcal{H} = \mathcal{H}_{\text{even}} \otimes \mathcal{H}_{\text{s.p.}}, \tag{1.1}$$

where $\mathcal{H}_{\text{even}}$ denotes the Hilbert space of collective states in the neighboring even-even system, and $\mathcal{H}_{\text{s.p.}}$ the Hilbert space of single-particle states. The particle-plus-rotor model [1] constitutes a classical example of a model constructed in such a way. The Hilbert space $\mathcal{H}_{\text{even}}$ is constructed in this case as a model space of a rotor without any explicit reference

1

to a microscopic description of states in an even-even core. The Pauli correlations between the odd fermion and the fermions comprising the even-even core are thus simply neglected. A similar approximation is also made in the quasiparticle-plus-core model [2], where pairing correlations are taken into account by considering in $\hat{\mathcal{H}}_{\text{s.p.}}$ quasiparticles instead of particles, and in $\hat{\mathcal{H}}_{\text{even}}$ both neighboring even-even cores.

In the phenomenological models of such odd fermion systems the Hamiltonian is assumed to be of the form

$$\hat{H} = \hat{H}_{\text{even}} + \hat{H}_{\text{s.p.}} + \hat{H}_{\text{int}}, \tag{1.2}$$

where the three components describe the even-even core, the single-particle states, and the interaction between them, respectively. Although the interaction mixes the eigenstates of $\hat{H}_{\text{even}} + \hat{H}_{\text{s.p.}}$, it is usually introduced to describe dynamical effects rather than corrections induced by the neglect of Pauli correlations in the basis states of the full $\mathcal{H}$.

It is worthwhile to recall here that even a Slater-determinant wave function of an odd nucleus,

$$\Psi(x_1 \ldots x_A) = \tfrac{1}{\sqrt{A!}} \sum_P (-1)^P \psi_{i_1}(x_1) \ldots \psi_{i_A}(x_A), \tag{1.3}$$

(where P is a permutation of indices, $P(1, \ldots, A) = i_1, \ldots, i_A$, with $(-1)^P$ its parity), cannot be presented as a simple product of a single Slater determinant of the core and of a single odd-fermion wave function,

$$\Psi'(x_1 \ldots x_A) = \left[\tfrac{1}{\sqrt{(A-1)!}} \sum_{P'} (-1)^{P'} \psi_{i_1}(x_1) \ldots \psi_{i_{A-1}}(x_{A-1}) \right] \times \psi_A(x_A), \tag{1.4}$$

(where $P'(1, \ldots, A-1) = i_1, \ldots, i_{A-1}$). However, the Slater determinant of Eq. (1.3) belongs to the product Hilbert space (1.1), because it can be presented as a linear combination of states (1.4):

$$\Psi(x_1 \ldots x_A) = \sum_{j=1}^A (-1)^{j+1} \left[\tfrac{1}{\sqrt{(A-1)!}} \sum_{P'} (-1)^{P'} \psi_{j_1}(x_1) \ldots \psi_{j_{A-1}}(x_{A-1}) \right] \times \psi_j(x_A). \tag{1.5}$$

where the set of indices $j_1, \ldots, j_{A-1}$ comprises $1, \ldots, A$ with the index j excluded.

In principle, we can therefore think about restoring Pauli correlations by constructing an interaction $\hat{H}_{\text{int}}$ which would enforce or assure the mixing of states (1.4) in such a way as to obtain states (1.5). This task is virtually hopeless when the even states are described by a model which does not explicitly use fermion degrees of freedom. In the present study we consider and present relevant constructions when the core states are described by bosons which result from a rigorous boson (or boson-fermion) mapping. In this case it becomes possible to address Pauli correlations between a chosen core and surplus fermions in a systematic way.

A model for which such an analysis is of direct relevance is the interacting boson-fermion model (IBFM) [3], where Pauli correlations are at least partially accommodated on the phenomenological level through an exchange term which mimics the microscopic exchange of fermions between a single fermion and a fermion pair. (There is microscopic evidence that the fermion quadrupole pairing interaction may be largely responsible for such an exchange term in the IBFM ; see Ref. [4] and references therein.)

The general formalism of boson and boson-fermion mappings or realizations of Lie algebras (from a nuclear physics point of view) and their present status have recently been reviewed extensively by Klein and Marshalek [5]. Amongst the open problems identified in that review is the one discussed above, phrased in the terminology of generalized quantized Bogoliubov-Valatin (QBV) transformations, with a further systematization of such transformations envisaged. This refers precisely to an approach where only some collective pair degrees of freedom are earmarked for bosonization, while the remaining degrees of freedom are to be treated as ideal fermions, kinematically independent from the bosons.

QBV results which have so far been obtained pertain first to the full so$(2N+1)$ algebra where all fermion pairs are bosonized and only states with at most one odd fermion subsequently need to be considered in the product space (1.1) (Ref. [5] and references cited therein). When bosons are associated with correlated fermion pairs defined by some collective subalgebra (and the product space (1.1) is naturally expected to contain states with more than one odd fermion), QBV-type results have so far only been obtained for a lim-

ited number of low rank subalgebras, namely su(2) [6,7], su(3) [8], so(4) [9] and so(5) [10]. Furthermore these results have been obtained exclusively from algebraic considerations, as opposed to derivation via coherent states - the two main avenues which have been explored for the mapping of even fermion systems.

In review [5] algebraic considerations are mostly stressed, although it is appreciated that the coherent state approach has been instrumental in the historical development, while also appealing for the economy and elegance with which it leads to boson mappings and the rigorous systematization of various mappings and results. As an example of the utility of the coherent state approach, one may quote the natural appearance of the $\mathcal{R}$-projection which plays an important role in the identification of spurious states as has been known for some time [11] and also vividly demonstrated recently [12].

It is therefore to be expected that a coherent state approach to boson-fermion mappings of odd systems, and ultimately generalizations of the QBV transformation, will play an important complementary role to present results and endeavors which exploit algebraic methods.

In this paper we present the proper framework to address the above program, namely introduce the appropriate coherent states (supercoherent states) and report on some first results. We also comment briefly on some possible further developments and hurdles which will have to be overcome. The organization of the paper is then as follows: In Sec. II we give a *résumé* of the background to generalized QBV mappings, stressing the restrictions on states which are to be included in the physical subspace of the ideal space. We discuss the distinction between ideal fermions and ideal quasifermions which becomes important for a discussion of properties of the ideal space. Supercoherent states are introduced in Sec. III for the so($2N$) algebra. We also present there various similarity transformations and define the mapping projected onto the space with at most one ideal fermion. In Sec. IV we obtain mappings induced by supercoherent states defined in the collective space, and give some examples for this case in Sec. V. Sec. VI contains a discussion of what has been achieved and where future effort should be directed to obtain QBV-type mappings from

4

supercoherent states for collective spaces.

II. QUANTIZED BOGOLIUBOV-VALATIN MAPPING
AND STRUCTURE OF THE IDEAL SPACE

We introduce the concept of a boson-fermion mapping and its specialization to the quantized Bogoliubov-Valatin (QBV) transformation (and possible generalizations) in the simple setting of a single j-shell. Suppressing the index j, we introduce fermion creation and annihilation operators $a^\mu \equiv a_\mu^+$ and a_μ, respectively, where μ can take on $N = 2j + 1$ values.

The algebra of products

$$N_{\mu\nu} = a^\mu a_\nu, \tag{2.1}$$

$$A^{\mu\nu} = a^\mu a^\nu = (A_{\mu\nu})^+, \tag{2.2}$$

generates the orthogonal algebra so$(2N)$. If supplemented by all the commutators of single and bifermion operators and the commutator of the single fermion operators themselves, the corresponding algebra is so$(2N+1)$.

We remark here that alternative to supplementing the so$(2N)$ algebra in the above fashion, one could of course replace the commutators of single fermion operators by the perhaps more natural anti-commutators, leading to an equivalent algebraic structure which, however, will then not be an algebra any more, but rather a superalgebra. (This superalgebra has a rather simple structure as it can be obtained by supplementing the algebra with its trivial center, the identity.) To the extent that supercoherent states will be used to induce the above algebraic (or equivalently superalgebraic) structure in an ideal space, these induced relations will typically hold on the whole ideal space, whereas other relations in the original fermion space, such as e.g. the trivial operator equivalence between a bifermion operator and the product of two single fermion operators, will only hold on the physical subspace of the ideal space. (See also Sec. III.)

A mapping for the full so(2N+1) would entail the introduction of a boson (associated operators $B^{\mu\nu} \equiv B^\dagger_{\mu\nu}$ and $B_{\mu\nu}$) for each fermion pair with indices $\mu\nu$, together with kinematically independent ideal fermions or ideal quasifermions (associated operators $\alpha^\mu \equiv \alpha^\dagger_\mu$ and α_μ). (The distinction between ideal fermions and ideal quasifermions is linked to the algebraic structure associated with the corresponding operators, as elaborated below.) Kinematic independence dictates that boson and ideal (quasi)fermion operators commute,

$$[B^{\mu\nu}, \alpha^\theta] = [B^{\mu\nu}, \alpha_\theta] = 0 \tag{2.3}$$

with similar results for the conjugate combinations. Furthermore, the physical subspace in the so(2N+1) case contains states with one ideal fermion at most, since the bosons B above had been introduced to represent fermion pair degrees of freedom.

In the physically interesting case where a collective subalgebra of so(2N+1) exists, one is really only interested in bosonizing the corresponding collective fermion pair(s), while treating all remaining degrees of freedom as fermions. In this case the ideal space should therefore not be limited with respect to the number of ideal fermions.

In the familiar example of pairing in a single j-shell where a single collective boson, $B^\dagger$ (say), suffices to represent the collective fermion pair, one would naturally aim at a product space description in terms of basis states of the type $(B^\dagger)^n \alpha^{\mu_1} \alpha^{\mu_2} \ldots |0)$, where the operators α represent ideal (quasi)fermions.

We recall here that our approach to boson-fermion mappings resorts under what has broadly been termed [5] the Beliaev-Zelevinsky-Marshalek method in which a mapping of *operators* precedes a mapping of *states*. States are then mapped after an association of extreme weight states has been made, usually in the form $|0\rangle \longleftrightarrow |0)$, as we also do here. The fermion vacuum $|0\rangle$ is annihilated by all fermion annihilation operators, $a_\mu|0\rangle = 0$, while the ideal space vacuum is annihilated by all ideal (quasi)fermion and all boson annihilation operators, namely $\alpha_\mu|0) = B_{\mu\nu}|0) = 0$.

We now turn to the difference between ideal fermions and ideal quasifermions, the latter also often referred to simply as quasifermions [5]. This difference resides in the way in

which single fermion degrees of freedom in the ideal space take into account information about the existing or pre-chosen fermion pair – boson association [13,5]. It is instructive to illustrate this in the su(2) case where in the ideal space the single boson degree of freedom $B^\dagger$ represents the original correlated fermion pair A^+. Clearly a similar configuration of fermions in the ideal space will be redundant. To take this into account, the algebra of ideal space fermions may be modified by imposing the *operator* constraint [5]

$$\sum_{\mu>0} \alpha^\mu \alpha^{\bar\mu} = 0. \tag{2.4}$$

This results in a modification of the fermion algebra in the ideal space [5], in which case the corresponding fermion-like operators are referred to as (ideal) *quasi*-fermion operators.

Alternatively to this procedure it is possible to retain the usual algebra for the ideal fermions (hence the corresponding terminology) and to incorporate the implications of a pre-chosen fermion pair – boson association into the ideal space *images* of the original single fermion operators [13,7].

As may be expected intuitively and has been shown explicitly [13] in the case of mappings for so(2N+1), ideal fermions and quasifermions may be related on an operator level by showing that the ideal quasifermion operators have the form of the corresponding ideal fermion operator times a projection operator. We emphasize, however, that a similar relationship has not yet been identified in detail for any of the cases where a collective subalgebra dictates the bosons that appear in the ideal space.

We note here that in the standard phenomenological IBFM it is indeed ideal fermions (and not quasifermions) that enter the description. In microscopic analyses which address the link between phenomenological IBFM parameters and those of an underlying shell model, present discrepancies [14] between results obtained from a mapping in terms of ideal fermions [4,15] and one constructed in terms of quasifermions [14], must at least partially be ascribed to the different algebraic properties of ideal fermions and quasifermions.

In the sequel we develop our formalism only for ideal fermions which seem not only to be more naturally suited for incorporation into coherent states, but also closer to the

spirit in which odd fermions (with unaltered algebra) are introduced phenomenologically, as discussed above and in Sec. I.

To conclude this Section, we briefly mention an alternative approach to the same problem, albeit one which mainly focuses on different or complimentary aspects, namely vector coherent state theory (VCS) [16,17]. Although this approach also uses "intrinsic" degrees of freedom to account for the odd fermions (ideal (quasi)fermions above), these degrees of freedom are utilized much more indirectly than ideal (quasi)fermions and are only defined in terms of their (left) action on the vector coherent states, rather than through an explicit algebraic structure. Furthermore this approach has so far mostly been utilized in the context of explicit construction of matrices for irreducible representations. It has also proven to be a valuable formalism for identifying physical subspaces through what is termed K-matrix theory (see Ref. [17] and references therein).

Aspects of the relationship between the QBV and VCS approaches have recently been studied by Klein, Walet, Geyer and Hahne [10].

III. THE SO($2N$) BOSON-FERMION MAPPINGS

The so($2N$) algebra consists of all bifermion operators in a fermion Fock space built of N single-particle states, i.e., $a^\mu a^\nu, a_\nu a_\mu$, and $\frac{1}{2}\delta^\mu_\nu - a^\mu a_\nu$, where a^μ and a_μ denote fermion creation and annihilation operators, respectively, $a^\mu = (a_\mu)^+$. The so($2N$) superalgebra is obtained by adding to the so($2N$) algebra the single-fermion operators themselves. Their anticommutation relations,

$$\{a^\mu, a_\nu\} = \delta^\mu_\nu, \tag{3.1}$$

determine the commutation relations between the single-fermion and bifermion operators, as well as the so($2N$) commutation relations between the bifermion operators.

A. The so($2N$) supercoherent states

The so($2N$) supercoherent states can be defined as [18]

$$|C, \phi\rangle = \exp\left(\tfrac{1}{2}C_{\mu\nu}a^{\mu}a^{\nu} + \phi_{\mu}a^{\mu}\right)|0\rangle \tag{3.2}$$

with the usual summation convention applied, and $|0\rangle$ denoting the fermion vacuum. These supercoherent states depend on $N(N-1)/2$ complex numbers, $C_{\mu\nu}=-C_{\nu\mu} = (C^{\mu\nu})^*$, and on N complex Grassmann variables, $\{\phi_{\mu}, \phi_{\nu}\}=\{\phi_{\mu}, \phi^{\nu}\}=0$, $\phi_{\mu}=(\phi^{\mu})^*$, which *anticommute* with the fermion operators, $\{\phi_{\mu}, a_{\nu}\}=\{\phi_{\mu}, a^{\nu}\}=0$. The "bra" supercoherent state,

$$\langle C, \phi| = \langle 0|\exp\left(\tfrac{1}{2}C^{\mu\nu}a_{\nu}a_{\mu} + \phi^{\mu}a_{\mu}\right), \tag{3.3}$$

facilitates the construction of a functional representation of the fermion Fock space. To every many-fermion state $|\Psi\rangle$ one namely associates a function of variables $C^{\mu\nu}$ and ϕ^{μ} according to the simple prescription

$$|\Psi\rangle \longleftrightarrow f_{\Psi}(C, \phi) = \langle C, \phi|\Psi\rangle. \tag{3.4}$$

Let us now consider the superalgebra composed of $N(N-1)/2$ boson creation and annihilation operators, $B^{\mu\nu}$ and $B_{\mu\nu}$, $B^{\mu\nu}=-B^{\nu\mu} = (B_{\mu\nu})^{\dagger}$, and of N ideal fermion creation and annihilation operators α^{μ} and α_{μ}, $\alpha^{\mu}=(\alpha_{\mu})^{+}$, i.e.,

$$\begin{aligned}
[B_{\mu\nu}, B^{\theta\rho}] &= \delta^{\theta}_{\mu}\delta^{\rho}_{\nu} - \delta^{\theta}_{\nu}\delta^{\rho}_{\mu}, \\
[B_{\mu\nu}, \alpha^{\mu}] &= [B_{\mu\nu}, \alpha_{\mu}] = 0, \\
\{\alpha^{\mu}, \alpha_{\nu}\} &= \delta^{\mu}_{\nu}, \\
\{\alpha^{\mu}, \alpha^{\nu}\} &= 0.
\end{aligned} \tag{3.5}$$

We refer to α^{μ} as ideal fermions to distinguish them from real fermions a^{μ}. The appellation "ideal" serves as a reminder that the creation operators α^{μ} *commute* with the boson annihilation operators $B_{\mu\nu}$, cf. Eq. (2.3), as opposed to the real fermion creation operators a^{μ} which *do not* commute with pair annihilation operators $a_{\mu}a_{\nu}$.

The supercoherent state for the superalgebra (3.5),

$$|C, \phi) = \exp\left(\tfrac{1}{2}C_{\mu\nu}B^{\mu\nu} + \phi_\mu \alpha^\mu\right)|0),\tag{3.6}$$

where $|0)$ denotes the ideal boson-fermion vacuum, $B_{\mu\nu}|0)=\alpha_\mu|0)=0$, gives rise to a functional representation of the ideal boson-fermion states:

$$|\Psi) \longleftrightarrow f_\Psi(C, \phi) = (C, \phi|\Psi).\tag{3.7}$$

We apply the usual notation by denoting the real fermion states and the ideal boson-fermion states by angled and rounded brackets, respectively. By comparing Eqs. (3.4) and (3.7) we see that both real and ideal states are now represented as functions of variables $C^{\mu\nu}$ and ϕ^μ, which provides us with a powerful method of mapping real fermion states into the ideal boson-fermion states (cf. Ref. [19]). Indeed, we may define the boson-fermion image of a fermion state by requiring that their functional images are equal, i.e.,

$$|\Psi) \longleftrightarrow |\Psi\rangle\tag{3.8}$$

if

$$(C, \phi|\Psi) \equiv \langle C, \phi|\Psi\rangle.\tag{3.9}$$

B. The so($2N$) Usui operator

All subsequent constructions of mappings between fermion operators and functions of boson and ideal fermion operators can be carried out as indicated above. One can, however, avoid the functional representation as an intermediate step in the mapping procedure by alternatively considering the supercoherent-state-inspired generalized Usui operator (see also Refs. [20,23])

$$U = \langle 0| \exp\left(\tfrac{1}{2}B^{\mu\nu}a_\nu a_\mu + \alpha^\mu a_\mu\right)|0).\tag{3.10}$$

This operator transforms a real fermion state into an ideal boson-fermion state

$$|\Psi) = U|\Psi\rangle\tag{3.11}$$

10

in such a way that, Eq. (3.9) holds automatically. Note that in defining the generalized Usui operator as in Eq. (3.10) we imply that the ideal fermion operators α^μ and α_μ *anticommute* with the real fermion operators a^ν and a_ν. By using the Usui operator one effectively avoids dealing with Grassmann variables which have rather unconventional properties, especially when one concerns derivatives with respect to Grassmann variables. However, reference to the supercoherent state (3.6) and the functional images remains useful, as also becomes clear from the subsequent discussion. (In Appendix C we also give an explicit example of how functional images are utilized to derive operator mappings.)

The mapping between operators acting in the real and ideal spaces can thus be realized by exploiting the Usui operator (3.10). If for a real fermion operator $\hat{O}$ one can find an operator $\mathcal{O}$ acting in the ideal space such that

$$\mathcal{O}U = U\hat{O}, \tag{3.12}$$

we say that $\hat{O}$ is mapped to $\mathcal{O}$, i.e. $\mathcal{O}$ is the boson-fermion image of $\hat{O}$ under the mapping:

$$\mathcal{O} \longleftrightarrow \hat{O}. \tag{3.13}$$

Such a definition does not determine properties of $\mathcal{O}$ in the full ideal space, but only those pertinent to the so-called physical subspace which consists of images $U|\Psi\rangle$ of all real fermion states $|\Psi\rangle$. Therefore, in the full ideal space the boson-fermion image of a fermion operator is not unique.

In Appendix A we derive the following boson-fermion mapping of fermion and bifermion operators as determined by the Usui operator of Eq. (3.10):

$$a^\mu a^\nu \longleftrightarrow B^{\mu\nu} - B^{\mu\rho}B^{\nu\theta}B_{\rho\theta}$$
$$-B^{\mu\rho}\alpha^\nu\alpha_\rho + B^{\nu\rho}\alpha^\mu\alpha_\rho + \alpha^\mu\alpha^\nu, \tag{3.14a}$$

$$a^\mu a_\nu \longleftrightarrow B^{\mu\theta}B_{\nu\theta} + \alpha^\mu\alpha_\nu, \tag{3.14b}$$

$$a_\nu a_\mu \longleftrightarrow B_{\mu\nu}, \tag{3.14c}$$

$$a^\nu \longleftrightarrow \alpha^\nu + B^{\nu\rho}\alpha_\rho, \tag{3.14d}$$

$$a_\nu \longleftrightarrow \alpha_\nu. \tag{3.14e}$$

11

It should be stressed that once the Usui operator is defined, the mapping of operators is also uniquely defined through Eq. (3.12), and the mappings (3.14a)–(3.14e) result from a simple calculation.

The images of superalgebra generators, obtained by using the Usui operator (3.10), are by construction guaranteed to fulfil the (anti)commutation relations only in the physical space. However, in the functional representation, these images have a particularly simple form containing only first order differential operators. In the ideal space, this means that only a single boson (or a single fermion) annihilation operator appears in any of the images in Eqs. (3.14a)–(3.14e). Together with the fact that the single-boson and single-ideal-fermion states are indeed physical, this ensures that the mapped operators fulfil (anti)commutation relations in the entire ideal space (cf. discussion in Sec. 2 of Ref. [11]). Of course, this fact can also be checked *a posteriori* by explicitly verifying the so(2N) superalgebra (anti)commutation relations of the operators in Eqs. (3.14a)–(3.14e).

The latter fact ensures that the boson-fermion images of real fermion states do not depend on the way we group fermion operators before we construct ideal states by consecutively acting with operator images in the ideal space. For example, one may obtain the boson-fermion image of the state $a^\mu a^\nu |0\rangle$ either by acting with the image of $a^\mu a^\nu$, Eq. (3.14a), on the boson-fermion vacuum $|0\rangle$, or by acting twice with images of single-fermion operators, Eq. (3.14d). The final result is the same in both cases, and a similar conclusion also holds in more complicated cases.

On the other hand, there is no guarantee that the image of a product of real fermion operators is equal to the product of their images. In general, this equality does not hold in the operator sense, but of course it does when action on a physical state is considered.

One notes the appearance of the ideal fermion pair $\alpha^\mu \alpha^\nu$ in the mapping of the real fermion pair $a^\mu a^\nu$, Eq. (3.14a). Therefore, the zero-, one-, and two-fermion states have the following ideal boson-fermion images:

$$|0\rangle \longleftrightarrow |0\rangle, \tag{3.15a}$$

$$a^\nu |0\rangle \longleftrightarrow \alpha^\nu |0\rangle, \tag{3.15b}$$

$$a^\mu a^\nu |0\rangle \longleftrightarrow [B^{\mu\nu} + \alpha^\mu \alpha^\nu] |0\rangle. \tag{3.15c}$$

The real fermion pairs are thus mapped onto linear combinations of ideal bosons and ideal fermion pairs. The mapping faithfully reproduces the structure of the real fermion space, i.e., only the symmetric combinations, $B^{\mu\nu} + \alpha^\mu \alpha^\nu$, appear in the physical space, while the antisymmetric ones, $B^{\mu\nu} - \alpha^\mu \alpha^\nu$, belong to the unphysical space.

As discussed in Sec. I, the mapping of fermion states onto the ideal boson-fermion space aims at such a description of Pauli correlations between even core and an odd particle which avoids explicit antisymmetrization. From this point of view, the mapping in Eqs. (3.14a)–(3.14e) does not represent any gain with respect to the original fermion space. Images of even fermion states, obtained by acting on the vacuum with the images of $a^\mu a^\nu$, Eq. (3.14a), contain the ideal fermion pair $\alpha^\mu \alpha^\nu$, cf. Eq. (3.15c), and an explicit antisymmetrization with any odd ideal fermion is still required. This is not a satisfactory solution, because one would like to achieve a complete bosonization of the real even-fermion-number states, similarly as is the case for the usual Dyson mapping, where ideal fermions are not used. In the following sections we discuss methods of addressing this deficiency.

C. Similarity-transformed so(2N) boson-fermion mappings

By applying a similarity transformation $\mathcal{W}$ to all images of superalgebra generators, $\mathcal{O}' = \mathcal{W}^{-1} \mathcal{O} \mathcal{W}$, one obtains another possible mapping of the superalgebra in the ideal space. This corresponds to using a new Usui operator, $U' = \mathcal{W}U$, and the new physical space is then equal to the similarity transform of the original physical space, $|\Psi\rangle' = U'|\Psi\rangle = \mathcal{W}|\Psi\rangle$. A suitable choice of the similarity transformation may therefore change the composition and properties of the physical space, and lead to mappings with a structure closer to the structures envisaged in Sec. I. In what follows we particularly aim at removing the unwelcome term $\alpha^\mu \alpha^\nu$ through an appropriate similarity transformation.

The similarity transformation $\mathcal{W}$ can always be presented in the form of an exponent, $\mathcal{W} = e^{\mathcal{T}}$, and evaluated by applying the Baker-Campbell-Hausdorf formula,

$$\mathcal{W}^{-1}\mathcal{O}\mathcal{W} = e^{-\mathcal{T}}\mathcal{O}e^{\mathcal{T}}$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}[\mathcal{T}[\mathcal{T}\ldots[\mathcal{T},\mathcal{O}]\ldots]]_k, \tag{3.16}$$

where the multiple commutator has to be taken k times. The series is infinite unless the multiple commutator vanishes at some order. Since we would like to preserve the finiteness of the boson mapping, we will consider only such operators $\mathcal{T}$ which lead to finite series in Eq. (3.16). Below we present results for two specific operators $\mathcal{T}$, while some details of the derivation are given in Appendix B.

Let us first discuss the similarity transformation (3.16) with $\mathcal{T}$ given by

$$\mathcal{T} = \tfrac{1}{2}B^{\mu\nu}\alpha_\nu\alpha_\mu, \tag{3.17}$$

which, when applied to mapping (3.14a)–(3.14e), yields

$$a^\mu a^\nu \longleftrightarrow \alpha^\mu\alpha^\nu - B^{\mu\rho}B^{\nu\theta}B_{\rho\theta}, \tag{3.18a}$$

$$a^\mu a_\nu \longleftrightarrow \alpha^\mu\alpha_\nu + B^{\mu\theta}B_{\nu\theta}, \tag{3.18b}$$

$$a_\nu a_\mu \longleftrightarrow \alpha_\nu\alpha_\mu + B_{\mu\nu}, \tag{3.18c}$$

$$a^\nu \longleftrightarrow \alpha^\nu, \tag{3.18d}$$

$$a_\nu \longleftrightarrow \alpha_\nu. \tag{3.18e}$$

One can see that the effect obtained is exactly the opposite to the desired one. Namely, the boson-fermion image of the real fermion pair operator $a^\mu a^\nu$, Eq. (3.18a), creates solely the ideal fermion pairs $\alpha^\mu\alpha^\nu$ when acting on the ideal vacuum, and the bosons do not at all appear in the physical space. The mapping in Eqs. (3.18a)–(3.18e) simply replaces real fermions by the ideal ones, and is therefore useless for practical applications.

On the other hand, mapping (3.18a)–(3.18e) may serve for an explicit check of some properties of other similarity transformed images. For example, it trivially fulfils the so($2N$) superalgebra (anti)commutation relations in the whole ideal space. Also trivially, the image

14

of any many-fermion state $a^\mu a^\nu \dots a^\rho |0\rangle$ is always $\alpha^\mu \alpha^\nu \dots \alpha^\rho |0\rangle$), no matter in which way we group (or do not group) the fermion operators in pairs to use either the image of $a^\mu a^\nu$, Eq. (3.18a), or that of a^ν, Eq. (3.18d). Therefore, any similarity transformed mapping will also have these properties.

From the above result one can guess that the desired goal may be met by using the hermitian conjugate of the operator in Eq. (3.17) to construct the similarity transformation. In Appendix B we show that by transforming the mapping (3.14a)–(3.14e) with

$$\mathcal{T} = \mathcal{T}(\mathcal{X}) \quad \text{for} \quad \mathcal{X} = \tfrac{1}{2}\alpha^\mu \alpha^\nu B_{\mu\nu} \tag{3.19}$$

one obtains

$$a^\mu a^\nu \longleftrightarrow B^{\mu\nu} - B^{\mu\rho} B^{\nu\theta} B_{\rho\theta}$$
$$-B^{\mu\rho}\alpha^\nu\alpha_\rho + B^{\nu\rho}\alpha^\mu\alpha_\rho - 2\alpha^\mu\alpha^\nu \mathcal{T}'\mathcal{N}, \tag{3.20a}$$

$$a^\mu a_\nu \longleftrightarrow B^{\mu\theta}B_{\nu\theta} + \alpha^\mu\alpha_\nu, \tag{3.20b}$$

$$a_\nu a_\mu \longleftrightarrow B_{\mu\nu}, \tag{3.20c}$$

$$a^\nu \longleftrightarrow \alpha^\nu \mathcal{T}'(1-\mathcal{N}) - \alpha^\rho \mathcal{T}' B^{\nu\theta}B_{\rho\theta} + B^{\nu\rho}\alpha_\rho, \tag{3.20d}$$

$$a_\nu \longleftrightarrow \alpha_\nu + \alpha^\rho B_{\nu\rho}\mathcal{T}'. \tag{3.20e}$$

In these equations, $\mathcal{N}$ is the ideal fermion number operator,

$$\mathcal{N} = \alpha^\mu\alpha_\mu, \tag{3.21}$$

while $\mathcal{T}$ is an analytical function of $\mathcal{X}$,

$$\mathcal{T} = \sum_{k=0}^{\infty} \lambda_k \mathcal{X}^k, \tag{3.22}$$

whose first derivative $\mathcal{T}'$ obeys the Ricatti equation [21],

$$2\mathcal{X}(\mathcal{T}'' + \mathcal{T}'^2) + \mathcal{T}' - 1 = 0. \tag{3.23}$$

This particular Ricatti equation can be solved in a closed form, and one obtains

15

$$\begin{aligned}
\mathcal{T}' &= \left(\tanh \sqrt{2\mathcal{X}}\right)/\sqrt{2\mathcal{X}}, \\
\mathcal{T} &= \log \cosh \sqrt{2\mathcal{X}},
\end{aligned} \tag{3.24}$$

which gives the similarity transformation

$$\mathcal{W} = \cosh \sqrt{\alpha^\mu \alpha^\nu B_{\mu\nu}}. \tag{3.25}$$

The square roots of operators, which appear in the above expressions, only serve as a short-hand notation to describe the power series. In fact, all these series contain only even powers of the argument, and therefore are the series of powers of the $\mathcal{X}=\frac{1}{2}\alpha^\mu \alpha^\nu B_{\mu\nu}$ operator itself (without the square root). For example, the lowest-order terms of the operator $\mathcal{T}'$, which enters mapping (3.20a)–(3.20e), read

$$\mathcal{T}' = 1 - \tfrac{1}{3}\alpha^\mu \alpha^\nu B_{\mu\nu} + \tfrac{2}{15}\left(\alpha^\mu \alpha^\nu B_{\mu\nu}\right)^2 - \dots \tag{3.26}$$

The functions of $\mathcal{X}$ are in principle infinite power series. However, convergence problems for these functions never appear if one considers their action on ideal states with a given number of bosons. Indeed, an n-th power of $\mathcal{X}$ annihilates all boson states which have a boson number smaller then n. Therefore, the infinite power series can be cut off at the n-th term, whenever only such ideal states are considered.

From Eq. (3.20a) one sees that the even-fermion-number states are now entirely bosonized. This is so because the last three terms in this equation give a contribution only if an ideal fermion is already present, while they can be disregarded in a pure boson subspace. Therefore, the images of the even fermion states reduce to those given by the standard Dyson mapping [22], for which the mapping of the one- to four-fermion states has now the following explicit form:

$$|0\rangle \longleftrightarrow |0), \tag{3.27a}$$

$$a^\nu |0\rangle \longleftrightarrow \alpha^\nu |0), \tag{3.27b}$$

$$a^\mu a^\nu |0\rangle \longleftrightarrow B^{\mu\nu} |0), \tag{3.27c}$$

$$a^\lambda a^\mu a^\nu |0\rangle \longleftrightarrow \left[\alpha^\lambda B^{\mu\nu} - \alpha^\mu B^{\lambda\nu} + \alpha^\nu B^{\lambda\mu}\right] |0)$$

16

$$-2\alpha^\lambda \alpha^\mu \alpha^\nu |0), \tag{3.27d}$$

$$a^\kappa a^\lambda a^\mu a^\nu |0\rangle \longleftrightarrow \left[B^{\kappa\lambda} B^{\mu\nu} - B^{\kappa\mu} B^{\lambda\nu} \right.$$
$$\left. + B^{\kappa\nu} B^{\lambda\mu} \right] |0). \tag{3.27e}$$

When an odd fermion is added to an even fermion state, the last term in Eq. (3.20d) does not contribute and the first two terms create an odd ideal fermion. However, this odd ideal fermion is accompanied by a whole series of terms created by the operator $\mathcal{T}'$. Therefore, the image of an odd real fermion state is a mixture of one-, three-, five-, e.t.c. ideal fermion states. More precisely, the series continues till the number of ideal fermions reaches the number of real fermions in the odd state being mapped.

This is exemplified in the image of the three-fermion state, Eq. (3.27d), which contains a three-ideal-fermion component. On the other hand, the one-ideal-fermion component of this image is built as an antisymmetrized product of the ideal fermion and of the boson representing the even core. The structure of odd states with more particles is similar.

When a second odd fermion is added to an odd state, all ideal fermions disappear by automatically *recombining* to bosons. This is not at all evident when looking at the rather involved structure of the single-fermion image, Eq. (3.20d), which contains an infinite series of terms *creating* ideal fermions. However, the odd state is itself built as a series of terms with different ideal-fermion numbers. Both series conspire in such a way that the recombination mechanism is perfectly realized and the Pauli correlations exactly preserved.

The operator $\mathcal{T}'$ can thus be regarded as an operator responsible for the necessary antisymmetrization between ideal fermions and bosons. An approximate antisymmetrization can be achieved by neglecting higher order terms in the series expansion (3.26). When keeping terms up to the n-th order one assures a correct antisymmetrization of states with the number of bosons not greater then n.

The exact preservation of Pauli correlations have been achieved here at the expense of complicated images of operators. When the mapped operators (3.20a)–(3.20e) are e.g. applied to a (real) fermion Hamiltonian, one obtains its boson-fermion image which acquires

17

many-body terms. One may then separate terms into the boson-boson, fermion-fermion, and boson-fermion parts and therefore split the Hamiltonian into three parts, as in Eq. (1.2). The boson-fermion part then represents an interaction which enforces correct antisymmetrization between the even core and odd fermions.

The exact images of odd states, obtained in this section, are probably too complicated to be effectively used in practical calculations. Our ultimate goal which, in the context of the full so($2N+1$) algebra, was to describe odd states in a product space of a *single* ideal fermion and bosons, has not yet been met. On the other hand, we may split the boson-fermion images of odd states into components having different ideal-fermion numbers, and consider them separately. Since these components are all orthogonal one to another, the antisymmetry properties must be valid for every one of them. In this way we may consider the images of odd states *projected* on the single-ideal-fermion subspace as the result of the mapping. Such a projection is not, of course, a similarity transformation and some properties of the mapping may therefore be modified. We analyze these questions in the next section.

D. Projected so($2N$) boson-fermion mapping

Apart from the term $\alpha^\mu \alpha^\nu$ in Eq. (3.14a), the mapping of bifermion operators, Eqs. (3.14a)–(3.14c), is identical to that derived by Dönau and Janssen [23]. They have used the Usui operator which is a projection of that of Eq. (3.10) on the ideal space with at most one ideal fermion, i.e.,

$$U_{01} \equiv \mathcal{P}_{01} U$$

$$= \langle 0| \exp\left(\tfrac{1}{2} B^{\mu\nu} a_\nu a_\mu\right) \left(1 + \alpha^\mu a_\mu\right) |0), \tag{3.28}$$

where $\mathcal{P}_{01} = \mathcal{P}_0 + \mathcal{P}_1$, and $\mathcal{P}_0 = |0)(0|$ and $\mathcal{P}_1 = \alpha^\mu |0)(0|\alpha_\mu$ are projection operators on the vacuum and on the one-fermion ideal states, respectively. Such an Usui operator maps real fermion operators according to the prescription

$$\mathcal{O}_{01} U_{01} = U_{01} \hat{O}, \tag{3.29}$$

where the image of $\hat{O}$ is denoted by $\mathcal{O}_{01}$. Hence, the mapping of the so$(2N)$ superalgebra reads

$$a^\mu a^\nu \longleftrightarrow B^{\mu\nu} - B^{\mu\rho} B^{\nu\theta} B_{\rho\theta}$$

$$- B^{\mu\rho} \alpha^\nu \alpha_\rho + B^{\nu\rho} \alpha^\mu \alpha_\rho, \tag{3.30a}$$

$$a^\mu a_\nu \longleftrightarrow B^{\mu\theta} B_{\nu\theta} + \alpha^\mu \alpha_\nu, \tag{3.30b}$$

$$a_\nu a_\mu \longleftrightarrow B_{\mu\nu}, \tag{3.30c}$$

$$a^\nu \longleftrightarrow \left(\alpha^\nu - \alpha^\rho B^{\nu\theta} B_{\rho\theta} \right) \mathcal{Q} + B^{\nu\rho} \alpha_\rho, \tag{3.30d}$$

$$a_\nu \longleftrightarrow \alpha_\nu + \alpha^\rho B_{\nu\rho} \mathcal{Q}. \tag{3.30e}$$

Here $\mathcal{Q}$ denotes an arbitrary operator which conserves the vacuum and annihilates one-ideal-fermion states, i.e.,

$$\mathcal{Q} = \mathcal{P}_0 + \mathcal{Q}'(1 - \mathcal{P}_0)(1 - \mathcal{P}_1), \tag{3.31}$$

where $\mathcal{Q}'$ is arbitrary.

The images of the so$(2N)$ generators, Eqs. (3.30a)–(3.30e) can be derived in two ways. First, one may follow a direct and standard way (see Appendix C) of explicitly considering the projected Usui operator, Eq. (3.28). Second, one may perform a kind of projection of the similarity images $\mathcal{O}$, Eqs. (3.20a)–(3.20e), by using the equation

$$\mathcal{P}_{01} \mathcal{O} = \mathcal{O}_{01} \mathcal{P}_{01}, \tag{3.32}$$

to find $\mathcal{O}_{01}$.

Equation (3.32) results from the definitions of boson-fermion images, Eqs. (3.12) and (3.29), and the relationship between the corresponding Usui operators (3.28). In particular, it has the following solutions for ideal fermions in the similarity mapping:

$$
\begin{aligned}
\mathcal{P}_{01}\alpha^{\nu} &= \alpha^{\nu}\mathcal{Q}\mathcal{P}_{01} \ , \\
\mathcal{P}_{01}\alpha_{\nu} &= \alpha_{\nu}\mathcal{P}_{01} \ , \\
\mathcal{P}_{01}\alpha^{\mu}\alpha_{\nu} &= \alpha^{\mu}\alpha_{\nu}\mathcal{P}_{01} \ , \\
\mathcal{P}_{01}\mathcal{N} &= \mathcal{N}\mathcal{P}_{01} \ , \\
\mathcal{P}_{01}\mathcal{T}' &= \mathcal{P}_{01} \ , \\
\mathcal{P}_{01}\alpha^{\mu}\alpha^{\nu} &= 0 \ .
\end{aligned}
\tag{3.33}
$$

The mapping of the single-fermion operators, Eqs. (3.30d) and (3.30e), is the same as obtained by Geyer and Hahne [13], who have used for $\mathcal{Q}$ simply the vacuum projection operator, $\mathcal{Q}=\mathcal{P}_0$. Another possible choice is $\mathcal{Q}=1-\mathcal{N}$, where $\mathcal{N}$ is the ideal-fermion-number operator, Eq. (3.21). That $\mathcal{Q}$ is not unique, simply illustrates the fact that images of fermion operators in the ideal space are undetermined outside the physical space, which here consists only of zero- and one-ideal-fermion states. By the same token, the superalgebra (anti)commutation relations of the generator images in the ideal space, Eqs. (3.30a)–(3.30e), are fulfilled only in the physical space.

The mapping given in Eqs. (3.30a)–(3.30e) presents a satisfactory solution to the bosonization program presented in Sec. I. Starting from the vacuum $|0)$ the even fermion states are obtained by using the image of $a^{\mu}a^{\nu}$, Eq. (3.30a), and therefore are mapped on purely bosonic states. Then, the odd fermion is simply added on top of the bosonic state by using the image of single-fermion creation operator, Eq. (3.30d).

On the other hand, when an additional fermion is added to an odd-fermion state by acting again with the image of Eq. (3.30d), the presence of the projection operator $\mathcal{Q}$ assures that the odd fermion is annihilated and a boson created. This is a concrete realization of the recombination mechanism described in the previous section.

IV. BOSON-FERMION MAPPING

OF COLLECTIVE SPACE

In this section we concentrate our discussion on the collective subalgebra based on using the collective fermion-pair creation operators

$$A^i = \tfrac{1}{2}\chi^i_{\mu\nu}a^\mu a^\nu, \tag{4.1}$$

numbered by the collective index $i=1,\ldots,M$, where M is supposed to be much smaller than the number of all possible pairs $(N(N-1)/2)$. Together with the corresponding collective fermion-pair annihilation operators, $A_i=(A^i)^+$, all linearly independent commutators $[A_i, A^j]$, and the single-fermion operators, a^ν and a_ν, they are assumed to form a closed collective superalgebra. The closure conditions read

$$\begin{aligned}
\left[\left[A_i, A^j\right], A_k\right] &= c^{jl}_{ik}A_l, \\
\left[A^i, a_\nu\right] &= \chi^i_{\mu\nu}a^\mu, \\
\left[A^i, a^\nu\right] &= 0, \\
\{a^\mu, a_\nu\} &= \delta^\mu_\nu, \\
\{a^\mu, a^\nu\} &= 0,
\end{aligned} \tag{4.2}$$

where c^{jl}_{ik} are structure constants and the implicit summation over repeated collective index l is assumed.

The corresponding physical subspace of the ideal space is now envisaged to be comprised of ideal states with an arbitrary number of ideal fermions, of course still subject to reigning space limitations. Physically this reflects a description where only collective fermion pairs are bosonized, while all other fermion degrees of freedom are simply accommodated as ideal fermions.

Following Ref. [12], we assume that the collective pairs are orthogonal and normalized to a common number g, i.e.,

$$\langle 0|A_i A^j|0\rangle \equiv \tfrac{1}{2}\chi^{\mu\nu}_i \chi^j_{\mu\nu} = g\delta^j_i, \tag{4.3}$$

$(\chi^{\mu\nu}_i=(\chi^i_{\mu\nu})^*)$ which gives the commutation relation

$$\left[A^j, \left[A_i, A^l\right]\right] = c_{ik}^{jl} A^k \tag{4.4}$$

and the symmetry properties of structure constants

$$c_{ik}^{jl} = c_{ik}^{lj} = c_{ki}^{jl} = (c_{jl}^{ik})^*. \tag{4.5}$$

A. Collective supercoherent states

The collective supercoherent state is defined as

$$|C, \phi\rangle = \exp\left(C_i A^i + \phi_\mu a^\mu\right)|0\rangle \tag{4.6}$$

where $C_i = (C^i)^*$ are complex numbers and $\phi_\mu = (\phi^\mu)^*$ complex Grassmann variables, as described in Sec. III A. This supercoherent state suggests the collective Usui operator

$$U = \langle 0| \exp\left(B^i A_i + \alpha^\mu a_\mu\right)|0\rangle \tag{4.7}$$

which transforms collective even-fermion states, *and* collective states with added individual fermions, into an ideal space composed of collective bosons, $B^i = B_i^\dagger$, $[B^i, B_j] = \delta_j^i$, and of ideal fermions α^μ.

The mapping of operators can be obtained from the equation $\mathcal{O}U = U\hat{O}$, which gives the following mapping of the collective superalgebra, (4.2):

$$A^j \longleftrightarrow gB^j - \tfrac{1}{2} c_{ik}^{jl} B^i B^k B_l$$

$$\qquad\qquad - \chi_{\mu\rho}^j \chi_i^{\nu\rho} B^i \alpha^\mu \alpha_\nu + \tfrac{1}{2} \chi_{\mu\nu}^j \alpha^\mu \alpha^\nu, \tag{4.8a}$$

$$[A_i, A^j] \longleftrightarrow g\delta_i^j - c_{ik}^{jl} B^k B_l - \chi_{\mu\rho}^j \chi_i^{\nu\rho} \alpha^\mu \alpha_\nu, \tag{4.8b}$$

$$A_j \longleftrightarrow B_j, \tag{4.8c}$$

$$a^\nu \longleftrightarrow \alpha^\nu + \chi_i^{\nu\rho} B^i \alpha_\rho, \tag{4.8d}$$

$$a_\nu \longleftrightarrow \alpha_\nu. \tag{4.8e}$$

Defining the collective pairs of ideal fermions,

$$\mathcal{A}^j = \tfrac{1}{2}\chi^j_{\mu\nu}\alpha^\mu\alpha^\nu, \tag{4.9}$$

$\mathcal{A}_j = (\mathcal{A}^j)^\dagger$, one can present the above mapping in a form in which the pair amplitudes $\chi^j_{\mu\nu}$ do not appear explicitly:

$$A^j \longleftrightarrow \mathcal{A}^j - \tfrac{1}{2}c^{jl}_{ik}B^iB^kB_l + B^i[\mathcal{A}_i, \mathcal{A}^j], \tag{4.10a}$$

$$[A_i, A^j] \longleftrightarrow [\mathcal{A}_i, \mathcal{A}^j] - c^{jl}_{ik}B^kB_l, \tag{4.10b}$$

$$A_j \longleftrightarrow B_j, \tag{4.10c}$$

$$a^\nu \longleftrightarrow \alpha^\nu + B^i[\mathcal{A}_i, \alpha^\nu], \tag{4.10d}$$

$$a_\nu \longleftrightarrow \alpha_\nu. \tag{4.10e}$$

Similarly as in the so(2N) case, the image of the collective pair operator A^i, Eq. (4.10a), contains the corresponding ideal collective pair operator $\mathcal{A}^i$, and therefore the above mapping does not present any simplification in the description of Pauli correlations. In particular, the collective one-pair states are not bosonized, $A^i|0\rangle = (\mathcal{A}^i + B^i)|0\rangle$. In the following Section we again use a similarity transformation to remove the intruding term $\mathcal{A}^i$ from the image of A^i.

B. Similarity transformation of collective space

We begin the discussion of the similarity transformation (3.16) by showing that the $\mathcal{T}$ operator given by,

$$\mathcal{T} = B^i\mathcal{A}_i \tag{4.11}$$

leads to the mapping in which bosons and ideal fermions are entirely decoupled:

$$A^j \longleftrightarrow \mathcal{A}^j - \tfrac{1}{2}c^{jl}_{ik}B^iB^kB_l, \tag{4.12a}$$

$$[A_i, A^j] \longleftrightarrow [\mathcal{A}_i, \mathcal{A}^j] - c^{jl}_{ik}B^kB_l, \tag{4.12b}$$

$$A_j \longleftrightarrow \mathcal{A}_j + B_j, \tag{4.12c}$$

$$a^\nu \longleftrightarrow \alpha^\nu, \tag{4.12d}$$

$$a_\nu \longleftrightarrow \alpha_\nu, \tag{4.12e}$$

in analogy to the results obtained for the so(2N) superalgebra, Eqs. (3.18a)–(3.18e). However, the similarity transformation which is now responsible for removing the collective fermion pair $\mathcal{A}^j$ from the mapping of A^j, Eq. (4.10a), is more complicated than in the corresponding case of the so(2N) superalgebra. One has to consider the similarity transformation for

$$\mathcal{T} = \sum_{k=1}^{\infty} t^{i_1 \ldots i_k}_{j_1 \ldots j_k} \mathcal{A}^{j_1} \ldots \mathcal{A}^{j_k} B_{i_1} \ldots B_{i_k}, \tag{4.13}$$

where $t^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ is a totally symmetric tensor (in upper, as well as in lower indices) built from the structure constants c^{jl}_{ik}. This results in the mapping

$$A^j \longleftrightarrow gB^j - \tfrac{1}{2}c^{jl}_{ik}B^iB^kB_l - \left(B^i - \mathcal{A}^k\mathcal{T}'^i_{\ k}\right)\left(g\delta^j_i - [\mathcal{A}_i, \mathcal{A}^j]\right), \tag{4.14a}$$

$$[A_i, A^j] \longleftrightarrow [\mathcal{A}_i, \mathcal{A}^j] - c^{jl}_{ik}B^kB_l, \tag{4.14b}$$

$$A_j \longleftrightarrow B_j, \tag{4.14c}$$

$$a^\nu \longleftrightarrow \alpha^\nu + \left(B^i - \mathcal{A}^k\mathcal{T}'^i_{\ k}\right)\left([\mathcal{A}_i, \alpha^\nu] - [\mathcal{A}^l, [\mathcal{A}_i, \alpha^\nu]]B_j\mathcal{T}'^j_{\ l}\right), \tag{4.14d}$$

$$a_\nu \longleftrightarrow \alpha_\nu - [\mathcal{A}^k, \alpha_\nu]B_i\mathcal{T}'^i_{\ k}, \tag{4.14e}$$

provided the operators $\mathcal{T}'^i_{\ j}$ and $\mathcal{T}''^{im}_{\ jn}$,

$$\mathcal{T}'^i_{\ j} = \sum_{k=1}^{\infty} k\, t^{ii_1 \ldots i_{k-1}}_{jj_1 \ldots j_{k-1}} \mathcal{A}^{j_1} \ldots \mathcal{A}^{j_{k-1}} B_{i_1} \ldots B_{i_{k-1}}, \tag{4.15a}$$

$$\mathcal{T}''^{im}_{\ jn} = \sum_{k=1}^{\infty} k(k-1)\, t^{imi_1 \ldots i_{k-2}}_{jnj_1 \ldots j_{k-2}} \mathcal{A}^{j_1} \ldots \mathcal{A}^{j_{k-2}} B_{i_1} \ldots B_{i_{k-2}}, \tag{4.15b}$$

fulfil equations:

$$\left(\delta^j_l - g\mathcal{T}'^j_{\ l} + \frac{1}{2}\left(\mathcal{T}''^{kn}_{\ ml} + \mathcal{T}'^k_{\ m}\mathcal{T}'^n_{\ l}\right)c^{ji}_{kn}\mathcal{A}^m B_i\right)\mathcal{A}^l B_j = 0, \tag{4.16a}$$

$$\left(c^{im}_{kl}\mathcal{T}'^j_{\ m} - c^{ij}_{km}\mathcal{T}'^m_{\ l}\right)\mathcal{A}^l B_j = 0. \tag{4.16b}$$

Eqs. (4.16a) and (4.16b) represent recurrence relations for tensors $t^{i_1 \ldots i_k}_{j_1 \ldots j_k}$, which can be solved for particular structure constants c^{jl}_{ik}. Since the structure constants are not arbitrary matrices, but obey stringent conditions resulting from the Jacobi identities for the collective algebra, the recurrence relations cannot be solved unless these conditions are properly taken

24

into account. This is difficult without specifying a particular collective algebraic structure. Below we solve the recurrence relations for the unitary collective algebras.

The intruding term $\mathcal{A}^i$ is now absent from the mapping of A^i, Eq. (4.14a), and the even-fermion-number collective states are in fact entirely bosonized. This is so because the last term in (4.14a) vanishes when acting on a state where no ideal fermions exist,

$$\left(g\delta_i^j - [\mathcal{A}_i, \mathcal{A}^j]\right)|0\rangle = 0, \tag{4.17}$$

cf. Eq. (4.3).

Similarly as for the so($2N$) superalgebra, when an odd fermion is added to a collective even state, a series of terms appears in the ideal space. These terms have one, three, five... ideal bosons added to purely bosonic components.

When the next fermion is added to an odd state the ideal fermions will not in general disappear from the corresponding boson-fermion image. This reflects the fact that when two real arbitrary fermions are added to a collective even state, this state will not in general belong to the collective space of the next even nucleus. The collective superalgebra closure relations (4.2) do not ensure that the corresponding supergenerators leave the collective space invariant. This is obvious for the single-fermion creation operators a^μ, which create the complete fermion Fock space and therefore cannot conserve the collective space.

On the other hand, when two fermions are added to a collective even state, and the appropriate linear combination is then taken as in Eq. (4.1), so as to form a collective pair, the resulting state does belong to the collective space of the next even nucleus. If an analogous operation is performed in the ideal space, one observes the desired mechanism of a recombination of odd ideal fermions into bosons. More precisely, by acting on the series of terms which represents an odd ideal state with the series of terms (5.3d) which represents a single fermion operator, and next forming a collective pair (4.1), one sees that the two series conspire in such a way that ideal fermions disappear from the resulting expression.

The similarity mapping of the collective superalgebra faithfully represents properties of the underlying collective space. One obtains an exact description of Pauli correlations

between bosons representing collective even states and an odd fermion and repeated application of the images Eqs. (4.12d) and (4.12a) onto the ideal space vacuum will yield the physical subspace described below Eqs. (4.2).

V. EXAMPLES OF MAPPINGS
FOR COLLECTIVE SPACES

A. Similarity mapping for unitary superalgebra

Let us suppose that the collective operators form an $(\Omega + 1)$-dimensional symmetric representation of the unitary algebra su($l+1$), i.e., one has l collective pairs A_i. The simplest example is provided by the well-known quasispin su(2) algebra. By normalizing the collective pairs so that $g=\Omega$ one obtains Ω-independent structure constants:

$$c_{ik}^{jl} = \delta_i^j \delta_k^l + \delta_k^j \delta_i^l. \tag{5.1}$$

The recurrence relations can now be fulfilled by requiring that tensors $t_{j_1...j_k}^{i_1...i_k}$ are proportional to symmetrized products of the Kronecker delta's. This is equivalent to postulating the operator $\mathcal{T}$ to be a function of the operator $\mathcal{X}$, $\mathcal{T} = \mathcal{T}(\mathcal{X})$, where

$$\mathcal{X} = \mathcal{A}^i B_i \tag{5.2}$$

and leads to the following mapping:

$$A^j \longleftrightarrow B^j(\Omega - N_B) - \left(B^i - \mathcal{A}^i \mathcal{T}'\right)\left(\Omega \delta_i^j - [\mathcal{A}_i, \mathcal{A}^j]\right), \tag{5.3a}$$

$$[A_i, A^j] \longleftrightarrow \delta_i^j(\Omega - N_B) - B^j B_i - \left(\Omega \delta_i^j - [\mathcal{A}_i, \mathcal{A}^j]\right), \tag{5.3b}$$

$$A_j \longleftrightarrow B_j, \tag{5.3c}$$

$$a^\nu \longleftrightarrow \alpha^\nu + \left(B^i - \mathcal{A}^i \mathcal{T}'\right)\left([\mathcal{A}_i, \alpha^\nu] - [\mathcal{A}^j, [\mathcal{A}_i, \alpha^\nu]]B_j \mathcal{T}'\right), \tag{5.3d}$$

$$a_\nu \longleftrightarrow \alpha_\nu - [\mathcal{A}^i, \alpha_\nu]B_i \mathcal{T}', \tag{5.3e}$$

where $N_B=B^k B_k$ is the boson-number operator.

In these Equations, the operator $\mathcal{T}'$ obeys the Ricatti equation [21]

$$\mathcal{X}(\mathcal{T}'' + \mathcal{T}'^2) - \Omega\mathcal{T}' + 1 = 0. \tag{5.4}$$

Recalling that the number of bosons in the physical space is limited to Ω we have that $(\mathcal{X})^{\Omega+1} = 0$ and the solution can be postulated in the form of a polynomial,

$$\mathcal{T}' = \sum_{k=0}^{\Omega} \lambda'_k \mathcal{X}^k, \tag{5.5}$$

with the coefficients λ'_k determined from the recurrence relation

$$\lambda'_0 = \frac{1}{\Omega} \quad , \quad \lambda'_k = \frac{1}{\Omega - k + 1} \sum_{m=1}^{k} \lambda'_{m-1}\lambda'_{k-m}. \tag{5.6}$$

We see that the $(\Omega+1)$-th coefficient becomes singular, but this of course does not influence the solution (5.5). One also notes that for large Ω the series (5.5) is rapidly converging,

$$\mathcal{T}' = \frac{1}{\Omega} + \frac{\mathcal{X}}{\Omega^3} + \frac{2\mathcal{X}^2}{\Omega^4(\Omega - 1)} + \cdots \tag{5.7}$$

B. Similarity mapping for the quasispin

su(2) superalgebra

We conclude this section by specifying Eqs. (4.10a)–(4.10e) and (5.3a)–(5.3e) for the simplest case of the quasispin su(2) algebra composed of the single pair-creation operator $A^+ = \sum_{\mu>0} a^\mu a^{\bar{\mu}}$, its hermitian conjugate A which is the pair-annihilation operator, and of the fermion-number operator N. The boson-fermion mapping of Eqs. (4.10a)–(4.10e) then reads

$$A^+ \longleftrightarrow \Omega B^\dagger - B^\dagger B^\dagger B - B^\dagger \mathcal{N} + \mathcal{A}^\dagger, \tag{5.8a}$$

$$N \longleftrightarrow 2B^\dagger B + \mathcal{N}, \tag{5.8b}$$

$$A \longleftrightarrow B, \tag{5.8c}$$

$$a^\mu \longleftrightarrow \alpha^\mu + B^\dagger \alpha_{\bar{\mu}}, \tag{5.8d}$$

$$a_\mu \longleftrightarrow \alpha_\mu. \tag{5.8e}$$

27

The similarity transformation $e^{-\mathcal{T}}\mathcal{O}e^{\mathcal{T}}$, for $\mathcal{T}$ given by Eqs. (5.5) and (5.6) and $\mathcal{X}=\mathcal{A}^{\dagger}B$, removes the ideal fermion pair from the physical space of an even system:

$$A^{+} \longleftrightarrow \Omega B^{\dagger} - B^{\dagger}B^{\dagger}B - \left(B^{\dagger} - \mathcal{T}'\mathcal{A}^{\dagger}\right)\mathcal{N}, \tag{5.9a}$$

$$N \longleftrightarrow 2B^{\dagger}B + \mathcal{N}, \tag{5.9b}$$

$$A \longleftrightarrow B, \tag{5.9c}$$

$$a^{\mu} \longleftrightarrow \mathcal{T}'\left(\Omega - B^{\dagger}B\right)\alpha^{\mu} + \left(B^{\dagger} - \mathcal{T}'\mathcal{A}^{\dagger}\right)\alpha_{\bar{\mu}}, \tag{5.9d}$$

$$a_{\mu} \longleftrightarrow \alpha_{\mu} + \mathcal{T}'\alpha^{\bar{\mu}}B. \tag{5.9e}$$

Boson-fermion images of even and odd collective states have the following form

$$(A^{+})^{N}|0\rangle \longleftrightarrow \frac{\Omega!}{(\Omega - N)!}(B^{\dagger})^{N}|0\rangle, \tag{5.10a}$$

$$a^{\mu}(A^{+})^{N}|0\rangle \longleftrightarrow \frac{\Omega!}{(\Omega - N - 1)!}\alpha^{\mu}\mathcal{T}'(B^{\dagger})^{N}|0\rangle, \tag{5.10b}$$

where we see specifically that the single ideal fermion states ($N = 0$ in Eq. (5.10b)) are correctly normalized:

$$a^{\mu}|0\rangle \longleftrightarrow \alpha^{\mu}|0\rangle . \tag{5.11}$$

For the even non-collective states one finds e.g.

$$a^{\mu}a^{\bar{\mu}}(A^{+})^{N}|0\rangle \longleftrightarrow \frac{(\Omega - 1)!}{(\Omega - N - 1)!}(B^{\dagger})^{N+1}|0\rangle + \left[(\Omega - B^{\dagger}B)\mathcal{T}'^{2} + \mathcal{T}'\right]\left[\Omega\alpha^{\mu}\alpha^{\bar{\mu}} - \mathcal{A}^{\dagger}\right]\frac{(\Omega - 1)!}{(\Omega - N)!}(B^{\dagger})^{N}|0\rangle. \tag{5.12}$$

When Ω non-collective pairs $a^{\mu}a^{\bar{\mu}}$ are summed together to form the collective pair A^{+}, the second term in the image (5.12) vanishes because $\mathcal{A}^{\dagger} = \sum_{\mu>0}\alpha^{\mu}\alpha^{\bar{\mu}}$. The resulting image of the even collective state reduces to the state with $N+1$ collective bosons, as it should.

The images (5.9) bear a strong resemblance to similar results obtained in Ref. [7] and the two sets must in fact be related by a further similarity transformation which we have so far not been able to find.

VI. DISCUSSION

We have presented a framework which extends the construction of boson mappings through coherent states to the domain of boson-fermion mappings. This is accomplished by the introduction of Grassmann variables into supercoherent states. Calculations were facilitated by the identification and further use of the associated Usui operators.

The formalism allowed us to construct a known Dyson-type mapping for the full so($2N$+1) algebra, together with some other formal mappings not previously considered. We also obtained some first results for boson-fermion mappings relevant to collective subspaces. However, additional effort will have to be directed at this aspect on two levels.

On the formal level one may think in terms of solving the recurrence relations in Eqs. (4.16a) and (4.16b) for other examples than unitary algebras. Since low rank orthogonal algebras have played a prominent role in fermion models with dynamical symmetry, these seem to be of most immediate interest. Alternatively, or additionally, one may need innovation in either the construction of novel supercoherent states or appropriate similarity transformations. Furthermore, utilization of the results to make further contact between microscopic models and (semi-) phenomenological models such as the IBFM, is also called for.

APPENDIX A: BOSON-FERMION MAPPING OF THE

SO(2N) SUPERALGEBRA

In order to drive the boson-fermion mapping of the so($2N$) superalgebra, Eqs. (3.14a)–(3.14e), we use the standard method [19] of commuting real and ideal operators with the Usui operator (3.10). Let us denote by $\mathcal{U}$ the exponent appearing in the definition of the Usui operator,

$$\mathcal{U} = \exp\left(\mathcal{C}\right) = \exp\left(\tfrac{1}{2}B^{\mu\nu}a_\nu a_\mu + \alpha^\mu a_\mu\right), \tag{A1}$$

which acts in the product space of real and ideal states. We will apply to Eq. (3.12) two forms of the BCH formula (3.16),

$$\mathcal{O}\mathcal{U} = \mathcal{U}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}[\mathcal{C}[\mathcal{C}\ldots[\mathcal{C},\mathcal{O}]\ldots]]_k, \tag{A2a}$$

$$\mathcal{U}\hat{O} = \sum_{k=0}^{\infty}\frac{1}{k!}[\mathcal{C}[\mathcal{C}\ldots[\mathcal{C},\hat{O}]\ldots]]_k\mathcal{U}, \tag{A2b}$$

remembering that after the calculation of multiple commutators, $\mathcal{U}$ acts on the ideal (real) vacuum to the right (left).

We first consider the ideal fermion annihilation operator and Eq. (A2a),

$$\alpha_\nu\mathcal{U} = \mathcal{U}\left(\alpha_\nu - [\alpha^\mu a_\mu, \alpha_\nu]\right). \tag{A3}$$

The first term gives zero when acting to the right on the ideal vacuum, while the commutator reads

$$[\alpha^\mu a_\mu, \alpha_\nu] = -a_\nu \tag{A4}$$

(recall that ideal fermions anticommute with real fermions), whence the higher-order multiple commutators vanish. Therefore, one obtains

$$\alpha_\nu U = U a_\nu, \tag{A5}$$

i.e., the mapping (3.14e) is proved.

Second, we consider the real fermion creation operator and Eq. (A2b),

$$\mathcal{U}a^{\nu} = \left(a^{\nu} + \tfrac{1}{2}[B^{\mu\rho}a_{\rho}a_{\mu}, a^{\nu}] + [\alpha^{\mu}a_{\mu}, a^{\nu}]\right)\mathcal{U}. \tag{A6}$$

Again, the first term vanishes when acting to the left on the real vacuum, and the commutators read

$$\tfrac{1}{2}[B^{\mu\rho}a_{\rho}a_{\mu}, a^{\nu}] + [\alpha^{\mu}a_{\mu}, a^{\nu}] = B^{\nu\rho}a_{\rho} + \alpha^{\nu}. \tag{A7}$$

where both terms commute with $\mathcal{C}$ and $\mathcal{U}$. Using the previously derived Equation (A5) one finally has that

$$(B^{\nu\rho}\alpha_{\rho} + \alpha^{\nu})\,U = Ua^{\nu}, \tag{A8}$$

i.e., the mapping (3.14d) is proved.

Continuing similar derivations, one may consider $B_{\mu\nu}\mathcal{U}$ to prove mapping (3.14c), then $\mathcal{U}a^{\mu}a_{\nu}$ to prove (3.14b), and finally $\mathcal{U}a^{\mu}a^{\nu}$ to prove (3.14a).

APPENDIX B: SIMILARITY TRANSFORMATIONS IN THE IDEAL BOSON-FERMION SPACE

In order to derive the similarity-transformed boson-fermion images, Eqs. (3.18a)–(3.18e) and Eqs. (3.20a)–(3.20e), one first considers the multiple commutators in the BCH formula, Eq. (3.16), where the operator $\mathcal{T}$ is given as a power series (3.22) of $\mathcal{X}$. We will only consider such operators $\mathcal{X}$ and $\mathcal{O}$ that

$$[\mathcal{X}, [\mathcal{X}, \mathcal{O}]] = 0. \tag{B1}$$

In this case the commutator acts on a power series like a differentiation, i.e.

$$[\mathcal{X}^{k}, \mathcal{O}] = k\mathcal{X}^{k-1}[\mathcal{X}, \mathcal{O}] \tag{B2}$$

and

$$[\mathcal{T}, \mathcal{O}] = \mathcal{T}'[\mathcal{X}, \mathcal{O}]. \tag{B3}$$

31

Moreover, the multiple commutators vanish,

$$[\mathcal{T}, [\mathcal{T}, \mathcal{O}]] = 0, \tag{B4}$$

and the BCH formula reduces to

$$\mathcal{W}^{-1}\mathcal{O}\mathcal{W} = \mathcal{O} - \mathcal{T}'[\mathcal{X}, \mathcal{O}]. \tag{B5}$$

For the operator $\mathcal{T}$ given by Eq. (3.17) one therefore obtains the following similarity transformations:

$$
\begin{aligned}
\mathcal{W}^{-1} B^{\mu\nu} \mathcal{W} &= B^{\mu\nu}, \\
\mathcal{W}^{-1} B_{\mu\nu} \mathcal{W} &= B_{\mu\nu} + \alpha_\nu \alpha_\mu, \\
\mathcal{W}^{-1} \alpha^\nu \mathcal{W} &= \alpha^\nu - B^{\nu\rho} \alpha_\rho, \\
\mathcal{W}^{-1} \alpha_\nu \mathcal{W} &= \alpha_\nu,
\end{aligned} \tag{B6}
$$

which applied to the boson-fermion images in Eqs. (3.14a)–(3.14e) give those in Eqs. (3.18a)–(3.18e). Note that some products of ideal operators, like $\alpha^\mu \alpha^\nu$ for example, do not fulfil condition (B1). Their similarity transformation can, however, be calculated as products of transformations of separate factors, which do fulfil (B1).

When $\mathcal{T}$ is given as a power series in $\mathcal{X}$, Eq. (3.19), one has

$$
\begin{aligned}
\mathcal{W}^{-1} B^{\mu\nu} \mathcal{W} &= B^{\mu\nu} - \mathcal{T}' \alpha^\mu \alpha^\nu, \\
\mathcal{W}^{-1} B_{\mu\nu} \mathcal{W} &= B_{\mu\nu}, \\
\mathcal{W}^{-1} \alpha^\nu \mathcal{W} &= \alpha^\nu, \\
\mathcal{W}^{-1} \alpha_\nu \mathcal{W} &= \alpha_\nu - \mathcal{T}' B_{\mu\nu} \alpha^\mu,
\end{aligned} \tag{B7}
$$

and the mapping in Eqs. (3.20a)–(3.20e) is obtained by inserting these similarity transformations in Eqs. (3.14a)–(3.14e). For example, the similarity image of the single-fermion creation operator reads

$$a^\nu \longleftrightarrow \alpha^\nu + \left(B^{\nu\rho} - \mathcal{T}' \alpha^\nu \alpha^\rho \right) \left(\alpha_\rho - \mathcal{T}' B_{\theta\rho} \alpha^\theta \right). \tag{B8}$$

After normal-ordering and grouping together terms with α^ν one obtains

$$a^\nu \longleftrightarrow \alpha^\nu \left(1 - \mathcal{T}'\mathcal{N} - (\mathcal{T}'' + \mathcal{T}'^2)\alpha^\rho\alpha^\theta B_{\rho\theta}\right)$$

$$- \alpha^\theta \mathcal{T}' B^{\nu\rho} B_{\theta\rho} + B^{\nu\rho}\alpha_\rho. \tag{B9}$$

The term with second derivative $\mathcal{T}''$ appears as a result of commuting $B^{\nu\rho}$ and $\mathcal{T}'$. After using the Ricatti Equation (3.23) one obtains mapping (3.20d).

APPENDIX C: BOSON-FERMION MAPPING OF THE
SO($2N$) SUPERALGEBRA USING
PROJECTED SUPERCOHERENT STATES

Similarly as in Eq. (A1), we define the projected Usui operator in the product space as

$$\mathcal{U}_{01} = \exp(\mathcal{C})(1 + \alpha^\mu a_\mu) = (1 + \alpha^\mu a_\mu)\exp(\mathcal{C}) \tag{C1}$$

for

$$\mathcal{C} = \tfrac{1}{2}B^{\mu\nu}a_\nu a_\mu. \tag{C2}$$

Then we use the BCH formula to show that

$$a_\mu a_\nu \exp(\mathcal{C}) = B_{\nu\mu}\exp(\mathcal{C}). \tag{C3}$$

Considering the fermion annihilation operator one has

$$\mathcal{U}_{01}a_\nu = (a_\nu + \alpha^\mu a_\mu a_\nu)\exp(\mathcal{C}) \tag{C4}$$

and the pair of fermions in the second term can be replaced by a boson as in Eq. (C3), while the first term, when acting on the ideal vacuum, can be replaced by an ideal fermion, i.e.,

$$\mathcal{U}_{01}a_\nu|0\rangle = \alpha_\nu(1 + \alpha^\mu a_\mu)\exp\mathcal{C}|0\rangle + \alpha^\mu B_{\nu\mu}\exp\mathcal{C}|0\rangle. \tag{C5}$$

In order to obtain U_{01} in the second term on the right-hand side, we need to use the projection operator $\mathcal{Q}$, Eq. (3.31), which conserves the ideal vacuum and annihilates one-ideal-fermion states. Then one obtains

$$U_{01}a_\nu = (\alpha_\nu + \alpha^\mu B_{\nu\mu}\mathcal{Q})U_{01}, \tag{C6}$$

and mapping (3.30e) is proved.

Similarly, we use the BCH formula to show that

$$\exp(\mathcal{C})a^\nu = (a^\nu + B^{\nu\rho}a_\rho)\exp(\mathcal{C}). \tag{C7}$$

Considering the fermion creation operator one therefore has

$$\mathcal{U}_{01}a^\nu = (1 + \alpha^\mu a_\mu)(a^\nu + B^{\nu\rho}a_\rho)\exp(\mathcal{C}) \tag{C8}$$

and

$$\langle 0|\mathcal{U}_{01}a^\nu = \langle 0|\alpha^\nu \exp(\mathcal{C}) + B^{\nu\rho}\langle 0|\mathcal{U}_{01}a_\rho. \tag{C9}$$

Eq. (C6) can now be used to transform the second term, while the $\mathcal{Q}$ operator is again necessary to obtain U_{01} in the first term. Finally one obtains

$$U_{01}a^\nu = \alpha^\nu \mathcal{Q}U_{01} + (B^{\nu\rho}\alpha_\rho + B^{\nu\rho}\alpha^\mu B_{\nu\mu}\mathcal{Q})U_{01}, \tag{C10}$$

and mapping (3.30d) is proved.

One notes that the possibility to replace in Eq. (C4) an arbitrary fermion-pair annihilation operator by a boson-annihilation operator is the key element of the derivation. When considering collective algebras such a replacement is not possible, and therefore a projected mapping cannot be similarly derived in the collective space.

We conclude this appendix with an example of how functional images are directly utilized to derive operator mappings. The mapping (3.30e) is derived in this manner by defining a supercoherent state projected to a space with zero or one ideal fermions

$$\langle C, \phi| := \langle 0|(1 + \phi_\nu a_\nu)\exp(\tfrac{1}{2}C_{\mu\nu}a_\nu a_\mu)$$
$$\equiv \langle 0|(1 + \phi_\nu a_\nu)e^{\hat{C}}, \tag{C11}$$

similar to the state (3.3), except for the projection.

The image of a_μ relevant to the above space is now constructed as follows.

$$\langle C, \phi | a_\mu = \langle 0 | (1 + \phi_\nu a_\nu) a_\mu e^{\hat{C}}$$

$$= \langle 0 | (a_\mu + \phi_\nu a_\nu a_\mu) e^{\hat{C}}$$

$$= \langle 0 | (a_\mu + \phi_\nu \partial_{\mu\nu}) e^{\hat{C}}$$

$$= \partial_\mu \langle 0 | (1 + \phi_\mu a_\mu) e^{\hat{C}} + \langle 0 | \phi_\nu \partial_{\mu\nu)} e^{\hat{C}}$$

$$= (\partial_\mu + \phi_\nu \partial_{\mu\nu} \mathcal{Q}) \langle 0 | (1 + \phi_\mu a_\mu) e^{\hat{C}}$$

$$= (\partial_\mu + \phi_\nu \partial_{\mu\nu} \mathcal{Q}) \langle C, \phi |. \tag{C12}$$

In the second last line it is clear that the projector $\mathcal{Q}$ must enter in order to extract the supercoherent state required for the final operator association. This association is the standard one, namely that a Grassmann variable and its derivative are associated with, respectively, a (ideal) fermion creation and a (ideal) fermion annihilation operator, while the usual Bargmann representation for complex variables is used. From the (over-)completeness of coherent states one can now clearly extract from the result (C12) the *operator* equivalence (mapping) (3.30e).

REFERENCES

* Permanent address: Institute of Theoretical Physics, Warsaw University, Hoża 69, 00-681 Warsaw, Poland.

[1] A. Bohr and B. Mottelson, *Nuclear Structure*, (Benjamin, New York, 1975), Vol II.

[2] F. Dönau and S. Frauendorf, J. Phys. Soc. Japan **44**, 526 Suppl. (1978).

[3] F. Iachello and P. van Isacker, *The Interacting Boson-Fermion Model*, (Cambridge University Press, Cambridge, 1991).

[4] E.A. de Kock and H.B. Geyer, Phys. Rev. C **38**, 2887 (1988).

[5] A. Klein and E.R. Marshalek, Rev. Mod. Phys. **63**, 375 (1991).

[6] T. Suzuki and K. Matsuyanagi, Prog. Theor. Phys. **56**, 1156 (1976).

[7] H.B. Geyer and F.J.W. Hahne, S. Afr. J. Phys. **6**, 67 (1983).

[8] A. Klein and N.R. Walet, Nucl. Phys. **A515**, 207 (1990).

[9] F.J.W. Hahne and A. Klein, Phys. Lett. B **229**, 1 (1989).

[10] A. Klein, N.R. Walet, H.B. Geyer and F.J.W. Hahne in *Group Theory and Special Symmetries in Nuclear Physics*, Eds. J.P. Draayer and J. Jänecke (World Scientific, Singapore, 1992) 233-247

[11] J. Dobaczewski, Nucl. Phys. **A369**, 237 (1981).

[12] J. Dobaczewski, H.B. Geyer, and F.J.W. Hahne, Phys. Rev. C **44**, 1030 (1991).

[13] H.B. Geyer and F.J.W. Hahne, Phys. Lett. B **90**, 6 (1980).

[14] A. Klein and J.-Q. Chen, Phys. Rev. C **47**, February (1993).

[15] H.B. Geyer and I. Morrison, Phys. Rev. C **40**, 2383 (1989).

[16] K.T. Hecht, *The Vector Coherent State Method and its Application to Higher Symmetries*,

Lecture Notes in Physics, vol. **90** (Springer, New York, 1987).

[17] D.J. Rowe and J. Repka, J. Math. Phys. **32**, 2614 (1992).

[18] B.W. Fatyga, V.A. Kostelecký, M.M. Nieto and D.R. Truax, Phys. Rev. D **43**, 1403, (1991).

[19] J. Dobaczewski, Nucl. Phys. **A369**, 213 (1981).

[20] T. Usui Prog. Theor. Phys. **23**, 787 (1960).

[21] S. Hassani, *Foundations of Mathematical Physics*, (Allyn and Bacon, Boston, 1991), p 587

[22] D. Janssen, F. Dönau, S. Frauendorf, and R.V. Jolos, Nucl. Phys. **A172**, 145 (1971).

[23] F. Dönau and D. Janssen, Nucl. Phys. **A209**, 109 (1973).